Problem 8

Snow began to fall during the morning of February 2 and continued steadily into the afternoon. At noon a snowplow began removing snow from a road at a constant rate. The plow traveled 6 km from noon to 1 PM but only 3 km from 1 PM to 2 PM. When did the snow begin to fall? [*Hints:* To get started, let t be the time measured in hours after noon; let $x(t)$ be the distance traveled by the plow at time t; then the speed of the plow is dx/dt. Let b be the number of hours before noon that it began to snow. Find an expression for the height of the snow at time t. Then use the given information that the rate of removal R (in m^3/h) is constant.]

Solution

If snow falls steadily, then the rate the height of the snow increases in time must be constant (call it H). That is,

$$\frac{dh}{dt} = H.$$

If b represents the number of hours before noon it started snowing, then the expression for h is $h(t) = H(t + b)$. This gives the height of the snow at any time t hours past noon. Since velocity is the time derivative of position,

$$v(t) = \frac{dx}{dt},$$

the position, $x(t)$, can be obtained by integrating the velocity.

$$x(t) = \int v(t)\,dt$$

From noon to 1 PM, the snowplow travels 6000 m, and from 1 PM to 2 PM, the snowplow travels 3000 m. These two facts are expressed mathematically as

$$6000 = \int_0^1 v(t)\,dt$$
$$3000 = \int_1^2 v(t)\,dt.$$

Now we need to determine an expression that relates the velocity, the height of the snow, and the rate of snow removal. The volumetric flow rate is defined as

$$\frac{dV}{dt} = Av,$$

where dV/dt is the volume per unit time flowing through an area A with velocity v. The area is the height of the snow times the width of the snowplow, $h(t)w$, and dV/dt is R, a constant. So we have

$$R = wh(t)v(t)$$
$$R = wH(t + b)v(t)$$
$$v(t) = \frac{R}{wH(t + b)}.$$

Now that we know $v(t)$ in terms of b, we can use the two equations above to answer the question.

$$6000 = \int_0^1 \frac{R}{wH(t+b)}\,dt = \frac{R}{wH}\ln(t+b)\Big|_0^1$$
$$3000 = \int_1^2 \frac{R}{wH(t+b)}\,dt = \frac{R}{wH}\ln(t+b)\Big|_1^2$$

$$6000 = \frac{R}{wH}\ln\frac{b+1}{b}$$
$$3000 = \frac{R}{wH}\ln\frac{b+2}{b+1}$$

Dividing these two equations gives

$$2 = \frac{\ln\frac{b+1}{b}}{\ln\frac{b+2}{b+1}}.$$

$$2\ln\frac{b+2}{b+1} = \ln\frac{b+1}{b}$$
$$\left(\frac{b+2}{b+1}\right)^2 = \frac{b+1}{b}$$
$$\frac{b^2+4b+4}{b^2+2b+1} = \frac{b+1}{b}$$
$$\cancel{b^3} + 4b^2 + 4b = \cancel{b^3} + 2b^2 + b + b^2 + 2b + 1$$
$$4b^2 + 4b = 3b^2 + 3b + 1$$
$$b^2 + b = 1$$
$$b = \frac{-1 \pm \sqrt{5}}{2}$$

Since b has to be positive,

$$b = \frac{\sqrt{5}-1}{2} \approx 0.618 \text{ hours.}$$

If we multiply this by 60, we get the number of minutes prior to noon that it began to snow. $0.618 * 60 \approx 37$. Therefore, it started snowing at about 11:23 AM.